Aproximación de derivadas

Aproximaciones con mayor exactitud

Derivadas de orden superior

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Derivada Numérica (Repaso)

?

$$f'(x_0) \cong \frac{f(x_0+h)-f(x_0)}{h} \Longrightarrow E(h)$$

$$f'(x_0) \cong \frac{f(x_0)-f(x_0-h)}{h} \Longrightarrow E(h)$$

$$f'(x_0) \cong \frac{f(x_0+h)-f(x_0-h)}{2h} \Longrightarrow E(h^2)$$

Derivada Numérica (Repaso)

$$f'(x_0) \cong \frac{f(x_0+h)-f(x_0)}{h}$$

$$E(h) \cong \left| \frac{f''(x_0)}{2!} h \right|$$

$$f'(x_0) \cong \frac{f(x_0)-f(x_0-h)}{h}$$

$$f'(x_0) \cong \frac{f(x_0+h)-f(x_0-h)}{2h}$$

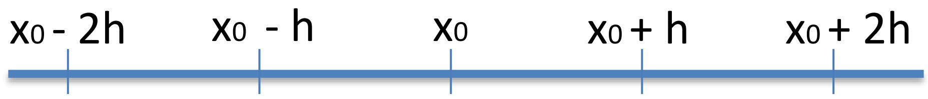
$$E(h^2) \cong \left| \frac{f'''(x_0)}{3!} h^2 \right|$$

$$f(x_0 + kh) = \sum_{n=0}^{\infty} k^n \frac{f^{(n)}(x_0)}{n!} h^n$$

$k > 0 \rightarrow$ Hacia delante

$k < 0 \rightarrow$ Hacia atrás

Aproximaciones con mayor exactitud



$$f(x_0-2h)=f(x_0)-f'(x_0)2h+\frac{f''(x_0)}{2!}4h^2-\frac{f'''(x_0)}{3!}8h^3+\frac{f''''(x_0)}{4!}16h^4-\frac{f'''''(x_0)}{5!}32h^5+\dots$$

$$8\left[f(x_0-h)=f(x_0)-f'(x_0)h+\frac{f''(x_0)}{2!}h^2-\frac{f'''(x_0)}{3!}h^3+\frac{f''''(x_0)}{4!}h^4-\frac{f'''''(x_0)}{5!}h^5+\dots\right]$$

$$8\left[f(x_0+h)=f(x_0)+f'(x_0)h+\frac{f''(x_0)}{2!}h^2+\frac{f'''(x_0)}{3!}h^3+\frac{f''''(x_0)}{4!}h^4+\frac{f'''''(x_0)}{5!}h^5+\dots\right]$$

$$f(x_0+2h)=f(x_0)+f'(x_0)2h+\frac{f''(x_0)}{2!}4h^2+\frac{f'''(x_0)}{3!}8h^3+\frac{f''''(x_0)}{4!}16h^4+\frac{f'''''(x_0)}{5!}32h^5+\dots$$

$$f(x_0+2h)+8f(x_0-h)-f(x_0-2h)-8f(x_0+h)=\dots$$

Aproximaciones con mayor exactitud

$$f(x_0+2h) = f(x_0) + 2f^{'}(x_0)h + 4\frac{f^{''}(x_0)}{2!}h^2 + 8\frac{f^{'''}(x_0)}{3!}h^3 + 16\frac{f^{''''}(x_0)}{4!}h^4 + 32\frac{f^{'''''}(x_0)}{5!}h^5 + \ldots$$

+

$$8f(x_0-h) = 8f(x_0) - 8f^{'}(x_0)h + 8\frac{f^{''}(x_0)}{2!}h^2 - 8\frac{f^{'''}(x_0)}{3!}h^3 + 8\frac{f^{''''}(x_0)}{4!}h^4 - 8\frac{f^{'''''}(x_0)}{5!}h^5 + \ldots$$

−

$$f(x_0-2h) = f(x_0) - 2f^{'}(x_0)h + 4\frac{f^{''}(x_0)}{2!}h^2 - 8\frac{f^{'''}(x_0)}{3!}h^3 + 16\frac{f^{''''}(x_0)}{4!}h^4 - 32\frac{f^{'''''}(x_0)}{5!}h^5 + \ldots$$

−

$$8f(x_0+h) = 8f(x_0) + 8f^{'}(x_0)h + 8\frac{f^{''}(x_0)}{2!}h^2 + 8\frac{f^{'''}(x_0)}{3!}h^3 + 8\frac{f^{''''}(x_0)}{4!}h^4 + 8\frac{f^{'''''}(x_0)}{5!}h^5 + \ldots$$

$$f(x_0+2h) + 8f(x_0-h) - f(x_0-2h) - 8f(x_0+h) = -12f^{'}(x_0)h + 48\frac{f^{'''''}(x_0)}{5!}h^5$$

$$f^{'}(x_0) = \frac{-f(x_0+2h) - 8f(x_0-h) + f(x_0-2h) + 8f(x_0+h)}{12h} + 4\frac{f^{'''''}(x_0)}{5!}h^4 + \ldots$$

Error de Truncamiento

Aproximaciones con mayor exactitud

$$f'(x_0) \cong \frac{-f(x_0+2h) - 8f(x_0-h) + f(x_0-2h) + 8f(x_0+h)}{12h}$$

$$E(h^4) \cong \left| 4 \frac{f'''''(x_0)}{5!} h^4 \right|$$

Error de Truncamiento

Reglas practicas para aproximar derivadas de cualquier orden

- Colocar las series de Taylor equiespaciadas una debajo de la otra.
- Mediante combinaciones lineales se deben eliminar todas las derivadas de orden menor (sin anular la de interés). En simultaneo se intenta minimizar el error, es decir eliminar los siguientes términos de la serie.
- Finalmente se despeja el objetivo y se obtiene la aproximación del error.

Ejemplo para f'' utilizando dos aproximaciones:

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 + \frac{f'''''(x_0)}{5!}h^5 + \ldots$$

$+$

$$f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2!}h^2 - \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 - \frac{f'''''(x_0)}{5!}h^5 + \ldots$$

$$f(x_0+h) + f(x_0-h) = 2f(x_0) + 2\frac{f''(x_0)}{2!}h^2 + 2\frac{f''''(x_0)}{4!}h^4 + 2\frac{f''''''(x_0)}{6!}h^6 + \ldots$$

$$f''(x_0) = \frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{h^2} - 2\frac{f''''(x_0)}{4!}h^2 - 2\frac{f''''''(x_0)}{6!}h^4 - \ldots$$

Error de Truncamiento

- Colocar las series de Taylor equiespaciadas una debajo de la otra.
- Mediante combinaciones lineales se deben eliminar todas las derivadas de orden menor (sin anular la de interés). En simultaneo se intenta minimizar el error, es decir eliminar los siguientes términos de la serie.
- Finalmente se despeja el objetivo y se obtiene la aproximación del error.

$$f''(x_0) \cong \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2}$$

$$E(h^2) \cong \left| 2\frac{f''''(x_0)}{4!}h^2 \right|$$

Ejemplo: f' con 4 hacia delante

$$48\quad \left[f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 + \frac{f'''''(x_0)}{5!}h^5 + \dots \right]$$

$$-36\quad \left[f(x_0+2h) = f(x_0) + f'(x_0)2h + \frac{f''(x_0)}{2!}4h^2 + \frac{f'''(x_0)}{3!}8h^3 + \frac{f''''(x_0)}{4!}16h^4 + \frac{f'''''(x_0)}{5!}32h^5 + \dots \right]$$

$$16\quad \left[f(x_0+3h) = f(x_0) + f'(x_0)3h + \frac{f''(x_0)}{2!}9h^2 + \frac{f'''(x_0)}{3!}27h^3 + \frac{f''''(x_0)}{4!}81h^4 + \frac{f'''''(x_0)}{5!}243h^5 + \dots \right]$$

$$-3\quad \left[f(x_0+4h) = f(x_0) + f'(x_0)4h + \frac{f''(x_0)}{2!}16h^2 + \frac{f'''(x_0)}{3!}64h^3 + \frac{f''''(x_0)}{4!}256h^4 + \frac{f'''''(x_0)}{5!}1024h^5 + \dots \right]$$

$$48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h) = 25f(x_0) + 12f'(x_0)h - 288\frac{f'''''(x_0)}{5!}h^5 + \dots$$

$$\boxed{f'(x_0) = \frac{48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h) - 25f(x_0)}{12h}} + 24\frac{f'''''(x_0)}{5!}h^4 + \dots$$

Ejemplo: f' con 1 atrás y 3 adelante

-3 $\left[f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2!}h^2 - \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 - \frac{f'''''(x_0)}{5!}h^5 + \ldots \right]$

18 $\left[f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 + \frac{f'''''(x_0)}{5!}h^5 + \ldots \right]$

-6 $\left[f(x_0+2h) = f(x_0) + f'(x_0)2h + \frac{f''(x_0)}{2!}4h^2 + \frac{f'''(x_0)}{3!}8h^3 + \frac{f''''(x_0)}{4!}16h^4 + \frac{f'''''(x_0)}{5!}32h^5 + \ldots \right]$

1 $\left[f(x_0+3h) = f(x_0) + f'(x_0)3h + \frac{f''(x_0)}{2!}9h^2 + \frac{f'''(x_0)}{3!}27h^3 + \frac{f''''(x_0)}{4!}81h^4 + \frac{f'''''(x_0)}{5!}243h^5 + \ldots \right]$

$$-3f(x_0-h) + 18f(x_0+h) - 6f(x_0+2h) + f(x_0+3h) = 10f(x_0) + 12f'(x_0)h + 72\frac{f'''''(x_0)}{5!}h^5 + \ldots$$

$$f'(x_0) = \frac{-3f(x_0-h) + 18f(x_0+h) - 6f(x_0+2h) + f(x_0+3h) - 10f(x_0)}{12h} - 6\frac{f'''''(x_0)}{5!}h^4 + \ldots$$

Ejemplo: f'' con 4 hacia delante

$$-104 \quad \left[f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 + \frac{f'''''(x_0)}{5!}h^5 + \dots \right]$$

$$114 \quad \left[f(x_0+2h) = f(x_0) + f'(x_0)2h + \frac{f''(x_0)}{2!}4h^2 + \frac{f'''(x_0)}{3!}8h^3 + \frac{f''''(x_0)}{4!}16h^4 + \frac{f'''''(x_0)}{5!}32h^5 + \dots \right]$$

$$-56 \quad \left[f(x_0+3h) = f(x_0) + f'(x_0)3h + \frac{f''(x_0)}{2!}9h^2 + \frac{f'''(x_0)}{3!}27h^3 + \frac{f''''(x_0)}{4!}81h^4 + \frac{f'''''(x_0)}{5!}243h^5 + \dots \right]$$

$$11 \quad \left[f(x_0+4h) = f(x_0) + f'(x_0)4h + \frac{f''(x_0)}{2!}16h^2 + \frac{f'''(x_0)}{3!}64h^3 + \frac{f''''(x_0)}{4!}256h^4 + \frac{f'''''(x_0)}{5!}1024h^5 + \dots \right]$$

$$-104f(x_0+h) + 114f(x_0+2h) - 56f(x_0+3h) + 11f(x_0+4h) = -35f(x_0) + 24\frac{f''(x_0)}{2!}h^2 + 1200\frac{f'''''(x_0)}{5!}h^5 + \dots$$

$$\boxed{f''(x_0) = \frac{-104f(x_0+h) + 114f(x_0+2h) - 56f(x_0+3h) + 11f(x_0+4h) + 35f(x_0)}{12h^2}} - 100\frac{f'''''(x_0)}{5!}h^3 + \dots$$

Ejemplo: f’’ con 1 atrás y 3 adelante

11 $\left\{ f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2!}h^2 - \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 - \frac{f'''''(x_0)}{5!}h^5 + \dots \right\}$

6 $\left\{ f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 + \frac{f'''''(x_0)}{5!}h^5 + \dots \right\}$

4 $\left\{ f(x_0+2h) = f(x_0) + f'(x_0)2h + \frac{f''(x_0)}{2!}4h^2 + \frac{f'''(x_0)}{3!}8h^3 + \frac{f''''(x_0)}{4!}16h^4 + \frac{f'''''(x_0)}{5!}32h^5 + \dots \right\}$

-1 $\left\{ f(x_0+3h) = f(x_0) + f'(x_0)3h + \frac{f''(x_0)}{2!}9h^2 + \frac{f'''(x_0)}{3!}27h^3 + \frac{f''''(x_0)}{4!}81h^4 + \frac{f'''''(x_0)}{5!}243h^5 + \dots \right\}$

$$11f(x_0-h) + 6f(x_0+h) + 4f(x_0+2h) - f(x_0+3h) = 20f(x_0) + 24\frac{f''(x_0)}{2!}h^2 - 120\frac{f'''''(x_0)}{5!}h^5 + \dots$$

$$f''(x_0) = \frac{11f(x_0-h) + 6f(x_0+h) + 4f(x_0+2h) - f(x_0+3h) - 20f(x_0)}{12h^2} + 10\frac{f'''''(x_0)}{5!}h^3 + \dots$$

Ejemplo: f'' con 2 atrás y 2 adelante

16 $\left[f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2!}h^2 - \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 - \frac{f'''''(x_0)}{5!}h^5 + \dots \right]$

-1 $\left[f(x_0-2h) = f(x_0) - f'(x_0)2h + \frac{f''(x_0)}{2!}4h^2 - \frac{f'''(x_0)}{3!}8h^3 + \frac{f''''(x_0)}{4!}16h^4 - \frac{f'''''(x_0)}{5!}32h^5 + \dots \right]$

16 $\left[f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 + \frac{f'''''(x_0)}{5!}h^5 + \dots \right]$

-1 $\left[f(x_0+2h) = f(x_0) + f'(x_0)2h + \frac{f''(x_0)}{2!}4h^2 + \frac{f'''(x_0)}{3!}8h^3 + \frac{f''''(x_0)}{4!}16h^4 + \frac{f'''''(x_0)}{5!}32h^5 + \dots \right]$

$$16f(x_0-h) - f(x_0-2h) + 16f(x_0+h) - f(x_0+2h) = 30f(x_0) + 24\frac{f''(x_0)}{2!}h^2 + \dots$$

$$f''(x_0) = \frac{16f(x_0-h) - f(x_0-2h) + 16f(x_0+h) - f(x_0+2h) - 30f(x_0)}{12h^2} + \dots$$

Derivada Numérica (Repaso)

$$f\left(x\right)=\ln x$$

- Estimar la derivada de *f(x)* en *x*=3 con un h=0.01
- Calcular el error exacto cometido utilizando la derivada analítica

$$f^{'}(x_0)\cong\frac{f(x_0+h)-f(x_0)}{h}$$

$$f^{'}(x_0)\cong\frac{f(x_0+h)-f(x_0-h)}{2h}$$

$$f^{'}(x_0)\cong\frac{-f(x_0+2h)-8f(x_0-h)+f(x_0-2h)+8f(x_0+h)}{12h}$$

Ejercicio

- Colocar las series de Taylor equiespaciadas una debajo de la otra.

- Mediante combinaciones lineales se deben eliminar todas las derivadas de orden menor (sin anular la de interés). En simultaneo se intenta minimizar el error, es decir eliminar los siguientes términos de la serie.

- Finalmente se despeja el objetivo y se obtiene la aproximación del error.

Obtener una aproximación de la derivada segunda a partir de dos puntos hacia delante (k=1 y k=2)
Hallar la estimación del error de truncamiento y compararla con el valor real para la función *f(x)=ln(x)* utilizando un incremento de h=0.01 en x=3

$$f(x_0 + kh) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(kh)^n$$

Matemática Superior Aplicada

Ejercicio

UTN

$$f(x_0+2h) = f(x_0) + 2f'(x_0)h + 4\frac{f''(x_0)}{2!}h^2 + 8\frac{f'''(x_0)}{3!}h^3 + 16\frac{f''''(x_0)}{4!}h^4 + 32\frac{f'''''(x_0)}{5!}h^5 + \ldots$$

$-$

$$2\left[f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 + \frac{f'''''(x_0)}{5!}h^5 + \ldots \right]$$

$$f(x_0+2h) - 2f(x_0+h) = -f(x_0) + 2\frac{f''(x_0)}{2!}h^2 + 6\frac{f'''(x_0)}{3!}h^3 + 14\frac{f''''(x_0)}{4!}h^4 + 30\frac{f'''''(x_0)}{5!}h^5 + \ldots$$

$$2\frac{f''(x_0)}{2!}h^2 = f(x_0+2h) - 2f(x_0+h) + f(x_0) - 6\frac{f'''(x_0)}{3!}h^3 - 14\frac{f''''(x_0)}{4!}h^4 - 30\frac{f'''''(x_0)}{5!}h^5 - \ldots$$

$$f''(x_0)h^2 = f(x_0+2h) - 2f(x_0+h) + f(x_0) - 6\frac{f'''(x_0)}{3!}h^3 - 14\frac{f''''(x_0)}{4!}h^4 - 30\frac{f'''''(x_0)}{5!}h^5 - \ldots$$

$$f''(x_0) = \frac{f(x_0+2h) - 2f(x_0+h) + f(x_0)}{h^2} \boxed{- 6\frac{f'''(x_0)}{3!}h - 14\frac{f''''(x_0)}{4!}h^2 - 30\frac{f'''''(x_0)}{5!}h^3 - \ldots}$$

Ejercicio

$$f''(x_0) = \frac{f(x_0+2h) - 2f(x_0+h) + f(x_0)}{h^2} - 6\frac{f'''(x_0)}{3!}h - 14\frac{f''''(x_0)}{4!}h^2 - 30\frac{f'''''(x_0)}{5!}h^3 - \ldots$$

$$f''(x_0) \cong \frac{f(x_0+2h) - 2f(x_0+h) + f(x_0)}{h^2}$$

$$E(h) \cong \left| 6\frac{f'''(x_0)}{3!}h \right|$$

Ejercicio

$$f(x)=\ln x \quad f'(x)=\frac{1}{x} \quad f''(x)=-\frac{1}{x^2} \quad f'''(x)=2\frac{1}{x^3}$$

$$f''(x_0)\cong\frac{f(x_0+2h)-2f(x_0+h)+f(x_0)}{h^2}$$

$$f''(\underbrace{3}_{x_0})\cong\frac{f(\underbrace{3}_{x_0}+2*\underbrace{0.01}_{h})-2f(\underbrace{3}_{x_0}+\underbrace{0.01}_{h})+f(\underbrace{3}_{x_0})}{0.01^2}$$

$$f''(3)\cong\frac{1.105257-2*1.101940+1.098612}{0.0001}$$

$$f''(3)\cong -0.110375$$

Ejercicio

$$f(x) = \ln x \quad f'(x) = \frac{1}{x} \quad f''(x) = -\frac{1}{x^2} \quad f'''(x) = 2\frac{1}{x^3}$$

$$f''(3) \cong -0.110375$$

$$f''(3) = -\frac{1}{3^2} = -0.111111$$

$$\varepsilon = \left| -0.111111 - \left(-0.110375\right) \right|$$

$$\varepsilon = 0.000736$$

Ejercicio

$$f''(x_0) = \frac{f(x_0+2h) - 2f(x_0+h) + f(x_0)}{h^2} + E(h)$$

$$E(h) \cong \left| 6\frac{f'''(x_0)}{3!} h \right|$$

$$f(x) = \ln x \quad f'(x) = \frac{1}{x} \quad f''(x) = -\frac{1}{x^2} \quad f'''(x) = 2\frac{1}{x^3}$$

$$E(h) \cong \left| 6\frac{2/27}{3!} 0.01 \right| = 0.000741$$

$$\varepsilon = 0.000736$$