Aproximación de derivadas

Aproximaciones con mayor exactitud

Derivadas de orden superior

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Derivada Numérica (Repaso)

?

$$f'(x_0) \cong \frac{f(x_0+h)-f(x_0)}{h} \Longrightarrow E(h)$$

$$f'(x_0) \cong \frac{f(x_0)-f(x_0-h)}{h} \Longrightarrow E(h)$$

$$f'(x_0) \cong \frac{f(x_0+h)-f(x_0-h)}{2h} \Longrightarrow E(h^2)$$

Derivada Numérica (Repaso)

$$f'(x_0) \cong \frac{f(x_0+h)-f(x_0)}{h}$$

$$E(h) \cong \left| \frac{f''(x_0)}{2!} h \right|$$

$$f'(x_0) \cong \frac{f(x_0)-f(x_0-h)}{h}$$

$$f'(x_0) \cong \frac{f(x_0+h)-f(x_0-h)}{2h}$$

$$E\left(h^2\right) \cong \left| \frac{f'''(x_0)}{3!} h^2 \right|$$

Derivada Numérica (Repaso)

$$f(x) = \ln x \quad f'(x) = \frac{1}{x} \quad f''(x) = -\frac{1}{x^2} \quad f'''(x) = 2\frac{1}{x^3}$$

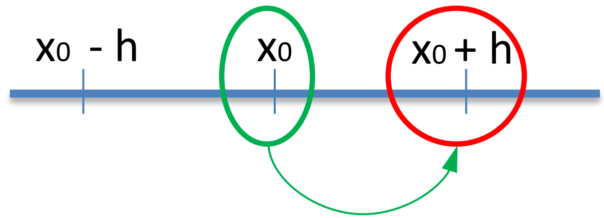
- Estimar la derivada de *f(x)* en *x*=3 con un h=0.01
- Estimar el error cometido
- Calcular el error exacto cometido utilizando la derivada analítica

$$f'(x_0) \cong \frac{f(x_0+h) - f(x_0)}{h} \qquad E(h) \cong \left| \frac{f''(x_0)}{2!} h \right|$$

$$f'(x_0) \cong \frac{f(x_0+h) - f(x_0-h)}{2h} \qquad E(h^2) \cong \left| \frac{f'''(x_0)}{3!} h^2 \right|$$

Serie de Taylor en torno a x_0

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$



$$f(x_0+h)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x_0+h-x_0)^n=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(h)^n$$

Taylor en torno a x_0 con intervalos equiespaciados

Hacia delante

$$f(x_0+h)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}h^n$$

k intervalos hacia delante

$$f(x_0+kh)=\sum_{n=0}^{\infty}k^n\frac{f^{(n)}(x_0)}{n!}h^n$$

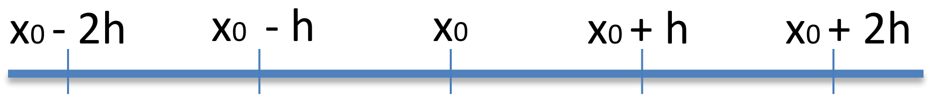
Hacia atrás

$$f(x_0-h)=\sum_{n=0}^{\infty}(-1)^n\frac{f^{(n)}(x_0)}{n!}h^n$$

k intervalos hacia atrás

$$f(x_0-kh)=\sum_{n=0}^{\infty}(-k)^n\frac{f^{(n)}(x_0)}{n!}h^n$$

Aproximaciones con mayor exactitud



$$f(x_0-2h)=f(x_0)-f'(x_0)2h+\frac{f''(x_0)}{2!}4h^2-\frac{f'''(x_0)}{3!}8h^3+\frac{f''''(x_0)}{4!}16h^4-\frac{f'''''(x_0)}{5!}32h^5+\ldots$$

$$8\left[f(x_0-h)=f(x_0)-f'(x_0)h+\frac{f''(x_0)}{2!}h^2-\frac{f'''(x_0)}{3!}h^3+\frac{f''''(x_0)}{4!}h^4-\frac{f'''''(x_0)}{5!}h^5+\ldots\right]$$

$$8\left[f(x_0+h)=f(x_0)+f'(x_0)h+\frac{f''(x_0)}{2!}h^2+\frac{f'''(x_0)}{3!}h^3+\frac{f''''(x_0)}{4!}h^4+\frac{f'''''(x_0)}{5!}h^5+\ldots\right]$$

$$f(x_0+2h)=f(x_0)+f'(x_0)2h+\frac{f''(x_0)}{2!}4h^2+\frac{f'''(x_0)}{3!}8h^3+\frac{f''''(x_0)}{4!}16h^4+\frac{f'''''(x_0)}{5!}32h^5+\ldots$$

$$f(x_0+2h)+8f(x_0-h)-f(x_0-2h)-8f(x_0+h)=\ldots.$$

Aproximaciones con mayor exactitud

$$f(x_0+2h)=f(x_0)+2f'(x_0)h+4\frac{f''(x_0)}{2!}h^2+8\frac{f'''(x_0)}{3!}h^3+16\frac{f''''(x_0)}{4!}h^4+32\frac{f'''''(x_0)}{5!}h^5+\ldots$$

+

$$8f(x_0-h)=8f(x_0)-8f'(x_0)h+8\frac{f''(x_0)}{2!}h^2-8\frac{f'''(x_0)}{3!}h^3+8\frac{f''''(x_0)}{4!}h^4-8\frac{f'''''(x_0)}{5!}h^5+\ldots$$

−

$$f(x_0-2h)=f(x_0)-2f'(x_0)h+4\frac{f''(x_0)}{2!}h^2-8\frac{f'''(x_0)}{3!}h^3+16\frac{f''''(x_0)}{4!}h^4-32\frac{f'''''(x_0)}{5!}h^5+\ldots$$

−

$$8f(x_0+h)=8f(x_0)+8f'(x_0)h+8\frac{f''(x_0)}{2!}h^2+8\frac{f'''(x_0)}{3!}h^3+8\frac{f''''(x_0)}{4!}h^4+8\frac{f'''''(x_0)}{5!}h^5+\ldots$$

$$f(x_0+2h)+8f(x_0-h)-f(x_0-2h)-8f(x_0+h)=-12f'(x_0)h+48\frac{f'''''(x_0)}{5!}h^5$$

$$f'(x_0)=\frac{-f(x_0+2h)-8f(x_0-h)+f(x_0-2h)+8f(x_0+h)}{12h}+4\frac{f'''''(x_0)}{5!}h^4+\ldots$$

Error de Truncamiento

Aproximaciones con mayor exactitud

$$f'(x_0) \cong \frac{-f(x_0+2h) - 8f(x_0-h) + f(x_0-2h) + 8f(x_0+h)}{12h}$$

$$E\left(h^4\right) \cong \left| 4\frac{f'''''(x_0)}{5!} h^4 \right|$$

Error de Truncamiento

Reglas practicas para aproximar derivadas de cualquier orden

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- Realizar las series de Taylor equiespaciadas y colocarlas una debajo de la otra.
- Mediante combinaciones lineales se deben eliminar todas las derivadas de orden menor (sin anular la de interés) y de manera simultanea intentar minimizar el error.
- Finalmente despejamos nuestro objetivo y obtenemos la aproximación del error.

Ejemplo para f'' utilizando dos aproximaciones:

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 + \frac{f'''''(x_0)}{5!}h^5 + \ldots$$

$+$

$$f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2!}h^2 - \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 - \frac{f'''''(x_0)}{5!}h^5 + \ldots$$

$$f(x_0+h) + f(x_0-h) = 2f(x_0) + 2\frac{f''(x_0)}{2!}h^2 + 2\frac{f''''(x_0)}{4!}h^4 + 2\frac{f''''''(x_0)}{6!}h^6 + \ldots$$

$$f''(x_0) = \frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{h^2} - 2\frac{f''''(x_0)}{4!}h^2 - 2\frac{f''''''(x_0)}{6!}h^4 - \ldots$$

Error de Truncamiento

- Realizar las series de Taylor equiespaciadas y colocarlas una debajo de la otra.
- Mediante combinaciones lineales se debe buscar eliminar todas las derivadas de orden menor sin anular la que buscamos aproximar.
- Finalmente despejamos nuestro objetivo y obtenemos la aproximación del error.

Ejemplo para f'' utilizando dos aproximaciones:

$$f''(x_0) \cong \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2}$$

$$E(h^2) \cong \left| 2\frac{f''''(x_0)}{4!}h^2 \right|$$

Ejercicio

- Realizar las series de Taylor equiespaciadas y colocarlas una debajo de la otra
- Mediante combinaciones lineales se debe buscar eliminar todas las derivadas de orden menor sin anular la que buscamos aproximar
- Finalmente despejamos nuestro objetivo y obtenemos la aproximación del error

Obtener una aproximación de la derivada segunda a partir de dos puntos hacia delante (k=1 y k=2)
Hallar la estimación del error de truncamiento y compararla con el valor real para la funcion *f(x)=ln(x)* utilizando un incremento de h=0.01 en x=3

$$f(x_0 + kh) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(kh)^n$$

Ejercicio

$$f(x_0+2h)=f(x_0)+2f'(x_0)h+4\frac{f''(x_0)}{2!}h^2+8\frac{f'''(x_0)}{3!}h^3+16\frac{f''''(x_0)}{4!}h^4+32\frac{f'''''(x_0)}{5!}h^5+\ldots$$

$-$

$$2\left[f(x_0+h)=f(x_0)+f'(x_0)h+\frac{f''(x_0)}{2!}h^2+\frac{f'''(x_0)}{3!}h^3+\frac{f''''(x_0)}{4!}h^4+\frac{f'''''(x_0)}{5!}h^5+\ldots\right]$$

$$f(x_0+2h)-2f(x_0+h)=-f(x_0)+2\frac{f''(x_0)}{2!}h^2+6\frac{f'''(x_0)}{3!}h^3+14\frac{f''''(x_0)}{4!}h^4+30\frac{f'''''(x_0)}{5!}h^5+\ldots$$

$$2\frac{f''(x_0)}{2!}h^2=f(x_0+2h)-2f(x_0+h)+f(x_0)-6\frac{f'''(x_0)}{3!}h^3-14\frac{f''''(x_0)}{4!}h^4-30\frac{f'''''(x_0)}{5!}h^5-\ldots$$

$$f''(x_0)h^2=f(x_0+2h)-2f(x_0+h)+f(x_0)-6\frac{f'''(x_0)}{3!}h^3-14\frac{f''''(x_0)}{4!}h^4-30\frac{f'''''(x_0)}{5!}h^5-\ldots$$

$$f''(x_0)=\frac{f(x_0+2h)-2f(x_0+h)+f(x_0)}{h^2}\boxed{-6\frac{f'''(x_0)}{3!}h-14\frac{f''''(x_0)}{4!}h^2-30\frac{f'''''(x_0)}{5!}h^3-\ldots}$$

Ejercicio

$$f''(x_0) = \frac{f(x_0+2h) - 2f(x_0+h) + f(x_0)}{h^2} - 6\frac{f'''(x_0)}{3!}h - 14\frac{f''''(x_0)}{4!}h^2 - 30\frac{f'''''(x_0)}{5!}h^3 - \ldots$$

$$f''(x_0) \cong \frac{f(x_0+2h) - 2f(x_0+h) + f(x_0)}{h^2}$$

$$E(h) \cong \left| 6\frac{f'''(x_0)}{3!}h \right|$$

Ejercicio

$$f(x) = \ln x \quad f'(x) = \frac{1}{x} \quad f''(x) = -\frac{1}{x^2} \quad f'''(x) = 2\frac{1}{x^3}$$

$$f''(x_0) \cong \frac{f(x_0 + 2h) - 2f(x_0 + h) + f(x_0)}{h^2}$$

$$f''(\overset{x_0}{3}) \cong \frac{f(\overset{x_0}{3} + 2 * 0.\overset{h}{0}1) - 2f(\overset{x_0}{3} + 0.\overset{h}{0}1) + f(\overset{x_0}{3})}{0.01^2}$$

$$f''(3) \cong \frac{1.105257 - 2 * 1.101940 + 1.098612}{0.0001}$$

$$f''(3) \cong -0.110375$$

Ejercicio

$$f(x) = \ln x \quad f'(x) = \frac{1}{x} \quad f''(x) = -\frac{1}{x^2} \quad f'''(x) = 2\frac{1}{x^3}$$

$$f''(3) \cong \text{-}0.110375$$

$$f''(3) = -\frac{1}{3^2} = -0.111111$$

$$\varepsilon = \left| -0.111111 - \left(-0.110375 \right) \right|$$

$$\varepsilon = 0.000736$$

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$$f''(x_0) = \frac{f(x_0+2h) - 2f(x_0+h) + f(x_0)}{h^2} + E(h)$$

$$E(h) \cong \left| 6\frac{f'''(x_0)}{3!} h \right|$$

$$f(x) = \ln x \quad f'(x) = \frac{1}{x} \quad f''(x) = -\frac{1}{x^2} \quad f'''(x) = 2\frac{1}{x^3}$$

$$E(h) \cong \left| 6\frac{2/27}{3!} 0.01 \right| = 0.000741$$

$$\varepsilon = 0.000736$$

Ejemplo: f' con 4 hacia delante

48 $\left\{ f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 + \frac{f'''''(x_0)}{5!}h^5 + \ldots \right\}$

-36 $\left\{ f(x_0+2h) = f(x_0) + f'(x_0)2h + \frac{f''(x_0)}{2!}4h^2 + \frac{f'''(x_0)}{3!}8h^3 + \frac{f''''(x_0)}{4!}16h^4 + \frac{f'''''(x_0)}{5!}32h^5 + \ldots \right\}$

16 $\left\{ f(x_0+3h) = f(x_0) + f'(x_0)3h + \frac{f''(x_0)}{2!}9h^2 + \frac{f'''(x_0)}{3!}27h^3 + \frac{f''''(x_0)}{4!}81h^4 + \frac{f'''''(x_0)}{5!}243h^5 + \ldots \right\}$

-3 $\left\{ f(x_0+4h) = f(x_0) + f'(x_0)4h + \frac{f''(x_0)}{2!}16h^2 + \frac{f'''(x_0)}{3!}64h^3 + \frac{f''''(x_0)}{4!}256h^4 + \frac{f'''''(x_0)}{5!}1024h^5 + \ldots \right\}$

$$48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h) = 25f(x_0) + 12f'(x_0)h - 288\frac{f'''''(x_0)}{5!}h^5 + \ldots$$

$$\boxed{f'(x_0) = \frac{48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h) - 25f(x_0)}{12h}} + 24\frac{f'''''(x_0)}{5!}h^4 + \ldots$$

Ejemplo: f’ con 1 atrás y 3 adelante

-3 $\left[f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2!}h^2 - \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 - \frac{f'''''(x_0)}{5!}h^5 + \ldots \right]$

18 $\left[f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 + \frac{f'''''(x_0)}{5!}h^5 + \ldots \right]$

-6 $\left[f(x_0+2h) = f(x_0) + f'(x_0)2h + \frac{f''(x_0)}{2!}4h^2 + \frac{f'''(x_0)}{3!}8h^3 + \frac{f''''(x_0)}{4!}16h^4 + \frac{f'''''(x_0)}{5!}32h^5 + \ldots \right]$

1 $\left[f(x_0+3h) = f(x_0) + f'(x_0)3h + \frac{f''(x_0)}{2!}9h^2 + \frac{f'''(x_0)}{3!}27h^3 + \frac{f''''(x_0)}{4!}81h^4 + \frac{f'''''(x_0)}{5!}243h^5 + \ldots \right]$

$$-3f(x_0-h) + 18f(x_0+h) - 6f(x_0+2h) + f(x_0+3h) = 10f(x_0) + 12f'(x_0)h + 72\frac{f'''''(x_0)}{5!}h^5 + \ldots$$

$$f'(x_0) = \frac{-3f(x_0-h) + 18f(x_0+h) - 6f(x_0+2h) + f(x_0+3h) - 10f(x_0)}{12h} - 6\frac{f'''''(x_0)}{5!}h^4 + \ldots$$

Ejemplo: f’’ con 4 hacia delante

-104 $$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 + \frac{f'''''(x_0)}{5!}h^5 + \ldots$$

114 $$f(x_0+2h) = f(x_0) + f'(x_0)2h + \frac{f''(x_0)}{2!}4h^2 + \frac{f'''(x_0)}{3!}8h^3 + \frac{f''''(x_0)}{4!}16h^4 + \frac{f'''''(x_0)}{5!}32h^5 + \ldots$$

-56 $$f(x_0+3h) = f(x_0) + f'(x_0)3h + \frac{f''(x_0)}{2!}9h^2 + \frac{f'''(x_0)}{3!}27h^3 + \frac{f''''(x_0)}{4!}81h^4 + \frac{f'''''(x_0)}{5!}243h^5 + \ldots$$

11 $$f(x_0+4h) = f(x_0) + f'(x_0)4h + \frac{f''(x_0)}{2!}16h^2 + \frac{f'''(x_0)}{3!}64h^3 + \frac{f''''(x_0)}{4!}256h^4 + \frac{f'''''(x_0)}{5!}1024h^5 + \ldots$$

$$-104f(x_0+h) + 114f(x_0+2h) - 56f(x_0+3h) + 11f(x_0+4h) = -35f(x_0) + 24\frac{f''(x_0)}{2!}h^2 + 1200\frac{f'''''(x_0)}{5!}h^5 + \ldots$$

$$f''(x_0) = \frac{-104f(x_0+h) + 114f(x_0+2h) - 56f(x_0+3h) + 11f(x_0+4h) + 35f(x_0)}{12h^2} - 100\frac{f'''''(x_0)}{5!}h^3 + \ldots$$

Ejemplo: f'' con 1 atrás y 3 adelante

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11 $\left[f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2!}h^2 - \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 - \frac{f'''''(x_0)}{5!}h^5 + \ldots \right]$

6 $\left[f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 + \frac{f'''''(x_0)}{5!}h^5 + \ldots \right]$

4 $\left[f(x_0+2h) = f(x_0) + f'(x_0)2h + \frac{f''(x_0)}{2!}4h^2 + \frac{f'''(x_0)}{3!}8h^3 + \frac{f''''(x_0)}{4!}16h^4 + \frac{f'''''(x_0)}{5!}32h^5 + \ldots \right]$

-1 $\left[f(x_0+3h) = f(x_0) + f'(x_0)3h + \frac{f''(x_0)}{2!}9h^2 + \frac{f'''(x_0)}{3!}27h^3 + \frac{f''''(x_0)}{4!}81h^4 + \frac{f'''''(x_0)}{5!}243h^5 + \ldots \right]$

$$11f(x_0-h) + 6f(x_0+h) + 4f(x_0+2h) - f(x_0+3h) = 20f(x_0) + 24\frac{f''(x_0)}{2!}h^2 - 120\frac{f'''''(x_0)}{5!}h^5 + \ldots$$

$$\boxed{f''(x_0) = \frac{11f(x_0-h) + 6f(x_0+h) + 4f(x_0+2h) - f(x_0+3h) - 20f(x_0)}{12h^2}} + 10\frac{f'''''(x_0)}{5!}h^3 + \ldots$$

Ejemplo: f'' con 2 atrás y 2 adelante

16 $\left[f(x_0 - h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2!}h^2 - \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 - \frac{f'''''(x_0)}{5!}h^5 + \ldots \right]$

-1 $\left[f(x_0 - 2h) = f(x_0) - f'(x_0)2h + \frac{f''(x_0)}{2!}4h^2 - \frac{f'''(x_0)}{3!}8h^3 + \frac{f''''(x_0)}{4!}16h^4 - \frac{f'''''(x_0)}{5!}32h^5 + \ldots \right]$

16 $\left[f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f''''(x_0)}{4!}h^4 + \frac{f'''''(x_0)}{5!}h^5 + \ldots \right]$

-1 $\left[f(x_0 + 2h) = f(x_0) + f'(x_0)2h + \frac{f''(x_0)}{2!}4h^2 + \frac{f'''(x_0)}{3!}8h^3 + \frac{f''''(x_0)}{4!}16h^4 + \frac{f'''''(x_0)}{5!}32h^5 + \ldots \right]$

$$16f(x_0 - h) - f(x_0 - 2h) + 16f(x_0 + h) - f(x_0 + 2h) = 30f(x_0) + 24\frac{f''(x_0)}{2!}h^2 + \ldots$$

$$f''(x_0) = \frac{16f(x_0 - h) - f(x_0 - 2h) + 16f(x_0 + h) - f(x_0 + 2h) - 30f(x_0)}{12h^2} + \ldots$$